

NON LINEAR SCHRÖDINGER EQUATION FOR THE TWISTED LAPLACIAN

P. K. RATNAKUMAR AND VIJAY KUMAR SOHANI

ABSTRACT. We establish the local well posedness of solution to the non-linear Schrödinger equation associated to the twisted Laplacian on \mathbb{C}^n in certain first order Sobolev space. Our approach is based on Strichartz type estimates, and is valid for a general class of non linearities including power type. The case $n = 1$ represents the magnetic Schrödinger equation in the plane with magnetic potential $A(z) = iz$, $z \in \mathbb{C}$.

1. INTRODUCTION

The free Schrödinger equation on \mathbb{R}^n is the PDE

$$i\partial_t \psi(x, t) + \Delta \psi(x, t) = 0, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}$$

which gives the quantum mechanical description of the evolution of a free particle in \mathbb{R}^n . If ψ is the solution of the Schrödinger equation, then $|\psi(x, t)|^2$ is interpreted as the probability density for finding the position of the particle in \mathbb{R}^n at a given time t . Let us consider the initial value problem

$$(1.1) \quad i\partial_t u(x, t) + \Delta u(x, t) = 0, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}$$

$$(1.2) \quad u(x, 0) = f(x).$$

For $f \in L^2(\mathbb{R}^n)$, the solution is given by the fourier transform:

$$u(x, t) = \int_{\mathbb{R}^n} e^{-it|\xi|^2} \hat{f}(\xi) e^{ix\xi} d\xi.$$

This may be written as

$$u(x, t) = e^{it\Delta} f(x)$$

interpreting the fourier inversion formula as the spectral decomposition in terms of the eigenfunctions of the Laplacian, see [15],[16].

More generally for any self adjoint differential operator L on \mathbb{R}^n , having the spectral representation $L = \int_E \lambda dP_\lambda$, we can associate the Schrödinger

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propagator $\{e^{-itL} : t \in \mathbb{R}\}$ given by

$$(1.3) \quad e^{-itL} f = \int_E e^{-it\lambda} dP_\lambda(f)$$

for $f \in L^2(\mathbb{R}^n)$. Here dP_λ denote the spectral projection for L . i.e., a projection valued measure supported on the spectrum E of L , see [2].

In this case, the function $u(x, t) = e^{-itL} f(x)$ solves the initial value problem for the Schrödinger equation for the operator L :

$$(1.4) \quad i\partial_t u(x, t) - Lu(x, t) = 0, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}$$

$$(1.5) \quad u(x, 0) = f(x)$$

with L now representing the corresponding Hamiltonian of the quantum mechanical system.

The significance this view point is that, most Hamiltonians of interest, namely the perturbation of the Laplacian with a potential V , (of the form $L = -\Delta + V$) or the magnetic Laplacian corresponding to the magnetic potential $(A_1(x), \dots, A_n(x))$ (of the form $L = \sum_{j=1}^n (i\partial_{x_j} + A_j(x))^2$) on \mathbb{R}^n , can be analysed with our approach, in terms of the spectral theory of the Hamiltonian.

In this paper, we concentrate on Schrödinger equation for an interesting magnetic Laplacian, namely the twisted Laplacian on \mathbb{C}^n ; also known as the special Hermite operator. The twisted Laplacian \mathcal{L} on \mathbb{C}^n is given by

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j)$$

where $Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2}\bar{z}_j$, $\bar{Z}_j = -\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2}z_j$, $j = 1, 2, \dots, n$. Here $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ denote the complex derivatives $\frac{\partial}{\partial x_j} \mp i\frac{\partial}{\partial y_j}$ respectively. The operator \mathcal{L} may be viewed as the complex analogue of the quantum harmonic oscillator Hamiltonian $H = -\Delta + |x|^2$ on \mathbb{R}^n , which has the representation

$$H = \frac{1}{2} \sum_{j=1}^n (A_j A_j^* + A_j^* A_j)$$

in terms of the creation operators $A_j = -\frac{d}{dx_j} + x_j$ and the annihilation operators $A_j^* = \frac{d}{dx_j} + x_j$, $j = 1, 2, \dots, n$. The operator \mathcal{L} was introduced by R. S. Strichartz [16], and called the special Hermite operator and it looks quite similar to the Hermite operator on \mathbb{C}^n : Infact

$$\mathcal{L} = -\Delta + \frac{1}{4}|z|^2 - i \sum_{j=1}^n \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right).$$

This may be re written as

$$\mathcal{L} = \sum_{j=1}^n \left[\left(i\partial_{x_j} + \frac{y_j}{2} \right)^2 + \left(i\partial_{y_j} - \frac{x_j}{2} \right)^2 \right]$$

which is of the form $\sum_{j=1}^{2n} [(i\partial_{w_j} - A_j(w))^2]$ hence represents a Schrödinger operator on \mathbb{C}^n for the magnetic vector potential $A(z) = iz, z \in \mathbb{C}^n$.

The Schrödinger equation for the magnetic potential with magnetic field decaying at infinity has been studied by many authors, see for instance Yajima [21], where he studies the propagator for the linear equation. In contrast, the nonlinear equation in our situation corresponds to a magnetic equation with a constant magnetic field, which has no decay. For more details on general magnetic Schrödinger equation corresponding to magnetic field without decay, see [1].

We consider the initial value problem for the non linear Schrödinger equation for the twisted Laplacian \mathcal{L} :

$$(1.6) \quad i\partial_t u(z, t) - \mathcal{L}u(z, t) = G(z, t, u), \quad z \in \mathbb{C}^n, \quad t \in \mathbb{R}$$

$$(1.7) \quad u(z, t_0) = f(z)$$

where G is a suitable C^1 function on $\mathbb{C}^n \times \mathbb{R} \times \mathbb{C}$. When $G \equiv 0$, and $f \in L^2(\mathbb{C}^n)$ the solution to this initial value problem is given by

$$u(z, t) = e^{-i(t-t_0)\mathcal{L}} f(z).$$

When $G(z, t, u) = g(z, t)$, the solution is given by the Duhamel formula

$$(1.8) \quad u(z, t) = e^{-i(t-t_0)\mathcal{L}} f(z) - i \int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z, s) ds.$$

Thus in the linear case, the solution is determined once the functions f and g are known.

For simplicity, we take $t_0 = 0$. The basic idea in the nonlinear analysis is the following heuristic reasoning based on the above formula. If the solution u is known, then one would expect u to satisfy the above equation with $g(z, s)$ replaced by $G(z, s, u(z, s))$:

$$(1.9) \quad u(z, t) = e^{-it\mathcal{L}} f(z) - i \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s, u(z, s)) ds.$$

Indeed one can show that u from a reasonable function space satisfies a PDE of the form (1.6), (1.7), if and only if u satisfies an integral equation of the form (1.9), see Lemma 5.1 for a precise version of this fact.

This reduces the existence theorem for the solution to the non linear Schrödinger equation to a fixed point theorem for the operator

$$(1.10) \quad \mathcal{H}(u)(z, t) = e^{-it\mathcal{L}}f(z) - i \int_0^t e^{-i(t-s)\mathcal{L}}G(z, s, u(z, s))ds$$

in a suitable subset of the relevant function space.

The non linearity G , that we consider is a C^1 function of the form

$$(1.11) \quad G(z, t, w) = \psi(x, y, t, |w|) w, \quad (x, y, t, w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{C},$$

where $z = x + iy \in \mathbb{C}^n, t \in \mathbb{R}, w \in \mathbb{C}$ and $\psi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times [0, \infty) \rightarrow \mathbb{C}$ is assumed to satisfy the following condition: The function $F = \psi, \partial_x \psi, \partial_y \psi$ and $|w| \partial_4 \psi(x, y, t, |w|)$, satisfy the inequality

$$(1.12) \quad |F(x, y, t, |w|)| \leq C|w|^\alpha,$$

for some constant C and $\alpha \in [0, \frac{2}{n-1})$.

The class of non-linearity given by (1.11), (1.12) includes in particular, power type non linearity of the form $|u|^\alpha u$. Moreover, the above class seems to be the most general form of non linearity adaptable to the Schrödinger equation for the special Hermite operator, for local existence via Kato's method [9]. The main difficulties in this approach is caused by the non commutativity of \mathcal{L} with $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ and the non compactibility of \mathcal{L} with the powertype non-linearity as observed in [4]. However we are able to overcome this difficulty by introducing the appropriate set of differential operators $L_j, M_j, j = 1, 2, \dots, n$ and working with a suitable Sobolev space defined using these operators, see sec 3.

We follow Kato's approach, using Strichartz estimates[15], as indicated above, to establish local existence. The advantage of Kato's method is that, it is useful, even when the conservation laws are not available. The main Strichartz estimate for the Schrödinger propagator for the twisted Laplacian (i.e., the special hermite operator) has already been proved in [13]. We also need some more relevant estimates like the associated retarded estimates etc, which we prove here, (Theorem 2.4).

There is a vast literature available for well posedness results for non linear Schrödinger equation on \mathbb{R}^n . See for instance the papers by Ginebre and Velo, [7], [8], Kato [9], the result of Cazenave and Weisler [5], the books by Cazenave [3] and Tao[20] and the extensive references there in. Some of the references that we came across dealing with magnetic Schrödinger equation are [21], [1] and [4] as mentioned before. In fact, the stability result discussed in [4], is actually the stability problem for the nonlinear Schrödinger equation for the twisted Laplacian in the plane.

Our main results in this paper is the well posedness for the non linear Schrödinger equation for the twisted Laplacian, in the Sobolev space $\tilde{W}^{1,2}(\mathbb{C}^n)$, (see sec 3 for the definition) and is given by the following two theorems:

Theorem 1.1. (Local existence) Assume that G is as in (1.11), (1.12) and $u(z, 0) = f(z) \in \tilde{W}^{1,2}(\mathbb{C}^n)$. Then there exist a number $T = T(\|u_0\|)$ such that the initial value problem (1.6), (1.7) has a unique solution $u \in C([-T, T]; \tilde{W}^{1,2}(\mathbb{C}^n))$.

Note that the interval $[-T, T]$ given by Theorem 1.1 need not be the maximal interval on which the solution exist. Now we discuss the uniqueness and stability in $\tilde{W}^{1,2}(\mathbb{C}^n)$ on the maximal interval denoted by (T_-, T_+) .

Definition 1.2. Let $\{f_m\}$ be any sequence such that $f_m \rightarrow f$ in $\tilde{W}^{1,2}(\mathbb{C}^n)$ and let u_m be the solution corresponding to the initial value f_m . We call the solution to the initial value problem (1.6)(1.7) stable in $\tilde{W}^{1,2}(\mathbb{C}^n)$ if $u_m \rightarrow u$ in $C(I, \tilde{W}^{1,2})$ for any compact interval $I \subset (T_-, T_+)$.

Theorem 1.3. The local solution established in Theorem 1.1 extends to a maximal interval (T_-, T_+) , where the following blowup alternative holds: either $|T_{\pm}| = \infty$ or $\lim_{t \rightarrow T_{\pm}} \|u(t, \cdot)\|_{\tilde{W}^{1,2}} = \infty$. More over, the solution is unique and stable in $C((T_-, T_+), \tilde{W}^{1,2})$.

Remark 1.4. In the definition of stability we can consider only the compact intervals and not the maximal interval. In fact u_m may not be defined on the maximal interval (T_-, T_+) for u . Also by blow up alternative u may not be in $L^\infty((T_-, T_+), \tilde{W}^{1,2})$.

The paper is organised as follows. In Section 2 we discuss the spectral theory of the twisted Laplacian \mathcal{L} and introduce the Schrödinger propagator $e^{it\mathcal{L}}$, and prove the relevant Strichartz estimates. In Section 3 we introduce certain first order Sobolev spaces associated to the twisted Laplacian, prove an embedding result in L^p spaces and other auxiliary estimates. The proofs of the main theorems are presented in section 4.

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2. SPECTRAL THEORY OF THE TWISTED LAPLACIAN

The twisted Laplacian is closely related to the sub Laplacian on the Heisenberg group, hence the spectral theory of this operator is closely connected

with the representation theory of the Heisenberg group. Here we give a brief review of the spectral theory of the twisted Laplacian \mathcal{L} . The references for the materials discussed in this section are the following books: Folland [6], and Thangavelu [17], [18].

The eigenfunctions of the operator \mathcal{L} are called the special Hermite functions, which are defined in terms of the Fourier-Wigner transform. For a pair of functions $f, g \in L^2(\mathbb{R}^n)$, the Fourier-Wigner transform is defined to be

$$V(f, g)(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f\left(\xi + \frac{y}{2}\right) \overline{g}\left(\xi - \frac{y}{2}\right) d\xi,$$

where $z = x + iy \in \mathbb{C}^n$. For any two multi-indices μ, ν the special Hermite functions $\Phi_{\mu\nu}$ are given by

$$\Phi_{\mu\nu}(z) = V(h_\mu, h_\nu)(z)$$

where h_μ and h_ν are Hermite functions on \mathbb{R}^n . Recall that for each nonnegative integer k , the one dimensional Hermite functions h_k are defined by

$$h_k(x) = \frac{(-1)^k}{\sqrt{2^k k! \sqrt{\pi}}} \left(\frac{d^k}{dx^k} e^{-x^2} \right) e^{\frac{x^2}{2}}.$$

Now for each multi index $\nu = (\nu_1, \dots, \nu_n)$, the n-dimensional Hermite functions are defined by the tensor product :

$$h_\nu(x) = \prod_{i=1}^n h_{\nu_i}(x_i), \quad x = (x_1, \dots, x_n).$$

A direct computation using the relations

$$\begin{aligned} \left(-\frac{d}{dx} + x \right) h_k(x) &= (2k+2)^{\frac{1}{2}} h_{k+1}(x), \\ \left(\frac{d}{dx} + x \right) h_k(x) &= (2k)^{\frac{1}{2}} h_{k-1}(x) \end{aligned}$$

satisfied by the Hermite functions h_k show that $\mathcal{L}\Phi_{\mu\nu} = (2|\nu| + n)\Phi_{\mu\nu}$. Hence $\Phi_{\mu,\nu}$ are eigenfunctions of \mathcal{L} with eigenvalue $2|\nu| + n$ and they also form a complete orthonormal system in $L^2(\mathbb{C}^n)$. Thus every $f \in L^2(\mathbb{C}^n)$ has the expansion

$$(2.1) \quad f = \sum_{\mu, \nu} \langle f, \Phi_{\mu\nu} \rangle \Phi_{\mu\nu}$$

in terms of the eigenfunctions of \mathcal{L} . The above expansion may be written as

$$(2.2) \quad f = \sum_{k=0}^{\infty} P_k f$$

where

$$(2.3) \quad P_k f = \sum_{\mu, |\nu|=k} \langle f, \Phi_{\mu, \nu} \rangle \Phi_{\mu \nu}$$

is the spectral projection corresponding to the eigenvalue $2k + n$. Now for any $f \in L^2(\mathbb{C}^n)$ such that $\mathcal{L}f \in L^2(\mathbb{C}^n)$, by self adjointness of \mathcal{L} , we have $P_k(\mathcal{L}f) = (2k + n)P_k f$. It follows that for $f \in L^2(\mathbb{C}^n)$ with $\mathcal{L}f \in L^2(\mathbb{C}^n)$

$$(2.4) \quad \mathcal{L}f = \sum_{k=0}^{\infty} (2k + n)P_k f.$$

Thus, we can define $e^{-it\mathcal{L}}$ as

$$(2.5) \quad e^{-it\mathcal{L}} f = \sum_{k=0}^{\infty} e^{-it(2k+n)} P_k f.$$

Note that $P_k f$ has the compact representation

$$P_k f(z) = (2\pi)^{-n} (f \times \varphi_k)(z)$$

in terms of the Laguerre function $\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}$, see [17]. Hence formally we can express $e^{it\mathcal{L}}$ as a twisted convolution operator:

$$e^{-it\mathcal{L}} f = f \times K_{it}$$

for $f \in \mathcal{S}(\mathbb{C}^n)$ where $K_{it}(z) = \frac{(4\pi i)^{-n}}{(\sin t)^n} e^{\frac{i(\cot t)|z|^2}{4}}$. Crucial to the local existence proof is a Strichartz type estimate for the Schrödinger propagator for the twisted Laplacian. We start with the following definition

Definition 2.1. Let $n \geq 1$. We say that a pair (q, p) is *admissible* if

$$2 < q < \infty \text{ and } \frac{1}{q} \geq n \left(\frac{1}{2} - \frac{1}{p} \right) \geq 0.$$

Remark 2.2. The admissibility condition on (q, p) implies that $2 \leq p < \frac{2n}{n-1}$.

The Strichartz type estimate for the Schrödinger propagator for the twisted Laplacian has been established in [13] and we state here a variant of the lemma on the convolution on the circle proved there. The proof follows exactly as in [13].

Lemma 2.3. : Let $K \in \text{weak } L^\rho([-a, a])$ for some $\rho > 1$ and let T be the operator given by

$$Tf(t) = \int_{[-a, a]} K(t-s)f(s)ds.$$

Then the following inequality holds

$$\|Tf\|_q \leq C_K \|f\|_{q'} \text{ for } q = 2\rho$$

with $C_K = C[K]_\rho$, where $[K]_\rho$ denotes the weak $L^\rho([-a, a])$ norm of K .

We use the compact notation $L_{[-T,T]}^{p,q}$ or simply $L^{p,q}$ for $L^q([-T, T], L^p(\mathbb{C}^n))$ and $L^{2,\infty}$ for the mixed L^p space $L^\infty([-T, T], L^2(\mathbb{C}^n))$. The main Strichartz type estimate we require is compiled in the following

Theorem 2.4. Let $f \in L^2(\mathbb{C}^n)$ and $g \in L^{q'}([-a, a]; L^{p'}(\mathbb{C}^n))$ where (q, p) denote an admissible pair with q' and p' denoting the corresponding conjugate indices. Then for $0 < a < \infty$, the following estimates hold over $[-a, a] \times \mathbb{C}^n$

$$(2.6) \quad \|e^{-it\mathcal{L}}f\|_{L^{p,q}} \leq C\|f\|_2$$

$$(2.7) \quad \left\| \int_0^t e^{-i(t-s)\mathcal{L}}g(z, s)ds \right\|_{L^{p,q}} \leq C\|g\|_{L^{p',q'}}$$

$$(2.8) \quad \left\| \int_0^t e^{-i(t-s)\mathcal{L}}g(z, s)ds \right\|_{L^{2,\infty}} \leq C\|g\|_{L^{p',q'}}$$

with a constant C is independent of f and g .

Proof. The proof of the estimate (2.6) given in [13] for $a = \pi$, works for any $a < \infty$, and relies on the following dispersive estimate for the complex semigroup $e^{-(r+it)\mathcal{L}}$ (see [13]):

$$(2.9) \quad \|e^{-(r+it)\mathcal{L}}f(z)\|_p \leq 2|\sin t|^{-2n(\frac{1}{2}-\frac{1}{p})}\|f\|_{p'}$$

valid for all $f \in L^p(\mathbb{C}^n)$, $2 \leq p \leq \infty$, $r > 0$ combined with a limiting argument as $r \rightarrow 0$.

Now we will give a direct proof of (2.9) for $r = 0$, essentially using the regularization argument used in [13], to deduce (2.7) and (2.9). The regularisation technique was first introduced in [11], [12], see also [14]. The results in these cases do not follow by the general Strichartz estimates established by Keel and Tao in [10] by lack of dispersive estimate for the kernel for the Schrödinger propagator in these cases.

For $f \in L^1 \cap L^2(\mathbb{C}^n)$, we have the isometry $\|e^{-it\mathcal{L}}f\|_2 = \|f\|_2$. Hence, using the series expansion for $e^{-(r+it)\mathcal{L}}f$ and appealing to the dominated convergence theorem for the sum, we can see that for any sequence $r_n \rightarrow 0$, $e^{-(r_n+it)\mathcal{L}}f \rightarrow e^{-it\mathcal{L}}f$ in $L^2(\mathbb{C}^n)$, it follows that $e^{-(r+it)\mathcal{L}}f(z) \rightarrow e^{-it\mathcal{L}}f(z)$ for almost all $z \in \mathbb{C}^n$ as $r \rightarrow 0$. This gives the inequality $\|e^{-it\mathcal{L}}f(z)\|_{L^\infty(\mathbb{C}^n)} \leq \frac{2}{|\sin t|^n}\|f\|_{L^1}$ from the corresponding inequality for the complex semigroup $e^{-(r+it)\mathcal{L}}$. Now the inequality (2.9) for $r = 0$ follows by interpolating this with the L^2 isometry.

The estimate (2.7) can be deduced from the inequality (2.9) with $f(z)$ replaced by $g(z, t)$ for each t , and the Lemma 2.3. In fact using Minkowski's

inequality for integrals and (2.9) with $r = 0$, we get

$$\left\| \int_0^t e^{-i(t-s)\mathcal{L}} g(z, s) ds \right\|_{L^p} \leq \int_0^a \frac{2}{|\sin(t-s)|^{2n(\frac{1}{2}-\frac{1}{p})}} \|g(\cdot, s)\|_{L^{p'}} ds.$$

Also for $p < \frac{2n}{n-1}$, we have $|\sin t|^{-2n(\frac{1}{2}-\frac{1}{p})} \in \text{weak } L^\rho[-a, a]$, for any $\rho \leq \frac{1}{2n(\frac{1}{2}-\frac{1}{p})}$.

Hence by Lemma 2.3 we see that

$$\left\| \int_0^t e^{-i(t-s)\mathcal{L}} g(z, s) ds \right\|_{L^{p,q}} \leq C \|g\|_{L^{p',q'}}$$

for any $q = 2\rho > 2$. Since $2\rho \leq \frac{1}{n(\frac{1}{2}-\frac{1}{p})}$, the above inequality is valid for any admissible pair, hence (2.7)

To prove (2.8) take $h \in \mathcal{S}(\mathbb{C}^n)$. By Hölder's inequality and estimate (2.6)

$$\begin{aligned} \left| \int_{\mathbb{C}^n} \int_0^t e^{i(t-s)\mathcal{L}} g(z, s) ds \overline{h(z)} dz \right| &\leq \int_{-a}^a \int_{\mathbb{C}^n} \left| g(z, s) \overline{e^{-i(t-s)\mathcal{L}} h(z)} \right| dz ds \\ &\leq \|g\|_{L^{p',q'}} \|e^{-i(t-s)\mathcal{L}} h(z)\|_{L^{p,q}} \\ &\leq C \|g\|_{L^{p',q'}} \|h(z)\|_2. \end{aligned}$$

Taking supremum over all h with $\|h\|_2 = 1$ we get the required estimate. \square

3. SOME AUXILIARY REGULARITY ESTIMATES

In this section we establish some auxiliary estimates used in the existence proof. The regularity of the solution is obtained through certain first order Sobolev space $\tilde{W}^{1,p}$ naturally associated to the one parameter group $\{e^{it\mathcal{L}} : t \in \mathbb{R}\}$, which we now introduce. Let L_j and M_j be the differential operators given by

$$L_j = \left(\frac{\partial}{\partial x_j} + i \frac{y_j}{2} \right), \text{ and } M_j = \left(\frac{\partial}{\partial y_j} - i \frac{x_j}{2} \right), \quad j = 1, 2, \dots, n$$

Definition 3.1. Let m be a non negative integer and $1 \leq p < \infty$. We say $f \in \tilde{W}^{m,p}(\mathbb{C}^n)$ if $S^\alpha f \in L^p(\mathbb{C}^n)$ for $|\alpha| \leq m$ where $S^\alpha = \prod_{i=1}^{2n} S_i^{\alpha_i}$ with $S_i = L_i$ for $1 \leq i \leq n$, $S_i = M_i$ for $n+1 \leq i \leq 2n$, and $|\alpha| = \alpha_1 + \dots + \alpha_{2n}$. $\tilde{W}^{m,p}(\mathbb{C}^n)$ is a Banach space with norm given by

$$\|f\|_{\tilde{W}^{m,p}} = \max\{\|S^\alpha f\|_{L^p} : |\alpha| \leq m\}.$$

In particular, $f \in \tilde{W}^{1,p}(\mathbb{C}^n)$ iff f , $L_j f$ and $M_j f$ are in $L^p(\mathbb{C}^n)$ for $1 \leq j \leq n$ and the norm in $\tilde{W}^{1,p}(\mathbb{C}^n)$ is given by

$$\|f\|_{\tilde{W}^{1,p}} = \max\{\|f\|_{L^p}, \|L_j f\|_{L^p}, \|M_j f\|_{L^p}, \quad j = 1, 2, \dots, n\}.$$

Remark 3.2. The differential operators L_j and M_j are the natural ones adaptable to the power type nonlinearity $G(u) = |u|^\alpha u$ and the generality that we consider here, see Lemma 3.5 and Proposition 3.7. The natural choice, namely the standard Sobolev space $W_{\mathcal{L}}^{1,p}(\mathbb{C}^n)$ defined using the twisted Laplacian \mathcal{L} (see [19]), is not suitable for treating such non linearities.

Remark 3.3. An interesting relation between the Sobolev space $\tilde{W}^{1,p}(\mathbb{C}^n)$ and the ordinary L^p Sobolev space $W^{1,p}(\mathbb{C}^n)$ is the following: If $u \in \tilde{W}^{m,p}(\mathbb{C}^n)$, then $|u| \in W^{m,p}(\mathbb{C}^n)$ for $m = 1$. This observation is crucial in the proof of the following embedding theorem for the Sobolev space $\tilde{W}^{1,2}$.

Lemma 3.4. [Sobolev Embedding Theorem] We have the continuous inclusion

$$\tilde{W}^{1,2} \hookrightarrow L^p(\mathbb{C}^n) \quad \begin{array}{ll} \text{for } 2 \leq p \leq \frac{2n}{n-1} & \text{if } n \geq 2, \\ \text{for } 2 \leq p < \infty & \text{if } n = 1. \end{array}$$

Proof. For $u \in \mathcal{S}(\mathbb{C}^n)$, we have

$$2|u| \frac{\partial}{\partial x_j} |u| = \frac{\partial}{\partial x_j} (\bar{u}u) = 2\Re \left(\bar{u} \frac{\partial}{\partial x_j} u \right) = 2\Re \left(\bar{u} \left(\frac{\partial}{\partial x_j} + \frac{iy_j}{2} \right) u \right).$$

Hence

$$\left| \frac{\partial}{\partial x_j} |u| \right| = \left| \Re \left(\frac{\bar{u}}{|u|} \left(\frac{\partial}{\partial x_j} + \frac{iy_j}{2} \right) u \right) \right| \leq |L_j(u)|$$

on the set $\{(z, t) \in \mathbb{C}^n \times \mathbb{R} \mid u(z, t) \neq 0\}$. Therefore, $\left| \frac{\partial}{\partial x_j} |u| \right| \leq |L_j u|$. Similarly we get $\left| \frac{\partial}{\partial y_j} |u| \right| \leq |M_j u|$. This implies that

$$\| |u| \|_{H^1} \leq (2n+1) \|u\|_{\tilde{W}^{1,2}}, \text{ for } u \in \mathcal{S}(\mathbb{C}^n).$$

Now by density of $\mathcal{S}(\mathbb{C}^n)$ in H^1 and $\tilde{W}^{1,2}$, the same inequality holds for all $u \in \tilde{W}^{1,2}$. Hence the result follows from the usual Sobolev embedding theorem on \mathbb{R}^{2n} . \square

Lemma 3.5. The operators L_j and M_j commutes with both the operators $e^{-it\mathcal{L}}$ and $\int_0^t e^{-i(t-s)\mathcal{L}} ds$, for $j = 1, 2, \dots, n$.

Proof. For $f \in C_c^\infty(\mathbb{C}^n)$, we have $e^{-it\mathcal{L}} f(z) = f \times K_{it}$, where K_{it} denotes the Schrödinger kernel: $K_{it}(z) = \frac{(4\pi i)^{-n}}{(\sin t)^n} e^{\frac{i \cot t}{4} |z|^2}$. A direct calculation shows that

$$\left(\partial_{x_j} + i \frac{y_j}{2} \right) \left[f(z-w) e^{\frac{i}{2} \Im(z \cdot \bar{w})} \right] = e^{\frac{i}{2} \Im(z \cdot \bar{w})} \left[\left(\partial_{w_j} + \frac{i}{2} v_j \right) f \right] (z-w)$$

$z = x + iy, w = u + iv$, from which the commutativity of L_j and $e^{-it\mathcal{L}}$ follows. A similar calculation also shows the commutativity of M_j and $e^{-it\mathcal{L}}$ for $j = 1, 2, \dots, n$ in $C_c^\infty(\mathbb{C}^n)$. From the above computation, it follows that the above

commutativity holds in $L^1_{loc}(\mathbb{C}^n)$, treating the locally integrable function as a distribution.

The commutativity of $\int_0^t e^{-i(t-s)\mathcal{L}}$ with L_j and M_j is also a consequence of the above observation. \square

For local existence, we need to establish the existence of a fixed point for the operator \mathcal{H} , defined by

$$\mathcal{H}(u)(z, t) = e^{-it\mathcal{L}}f(z) - i \int_0^t e^{-i(t-s)\mathcal{L}}G(z, s, u(z, s)) ds.$$

In the next two results, we prove some essential estimates required for this.

Lemma 3.6. Let $f \in \tilde{W}^{1,2}(\mathbb{C}^n)$. Then the following estimates hold:

$$(3.1) \quad \|e^{-it\mathcal{L}}f\|_{L^\infty([-T, T]; \tilde{W}^{1,2}(\mathbb{C}^n))} = \|f\|_{\tilde{W}^{1,2}(\mathbb{C}^n)}$$

$$(3.2) \quad \|e^{-it\mathcal{L}}f\|_{L^q([-T, T]; \tilde{W}^{1,p}(\mathbb{C}^n))} \leq C\|f\|_{\tilde{W}^{1,2}(\mathbb{C}^n)}$$

for admissible pairs (q, p) , with a constant C is independent of f .

Proof. Since both L_j and M_j commutes with the isometry $e^{-it\mathcal{L}}$, we have

$$\|Se^{-it\mathcal{L}}f\|_{L^2(\mathbb{C}^n)} = \|Sf\|_{L^2(\mathbb{C}^n)}$$

for every $t \in \mathbb{R}$ with $S = L_j$, or M_j , $j = 1, 2, \dots, n$ from which the equation (3.1) follows. Estimate (3.2) follows from the Strichartz type estimate (2.6) for $e^{it\mathcal{L}}$ using the above commutativity. \square

Proposition 3.7. Let $G(z, t, w)$ and α be as in (1.11), (1.12) and (q, p) an admissible pair with $p = \alpha + 2$, $q > 2$ and S as in Lemma 3.6. If $u \in L^\infty([-T, T]; \tilde{W}^{1,2}(\mathbb{C}^n)) \cap L^q([-T, T]; \tilde{W}^{1,p}(\mathbb{C}^n))$, then

$$G(z, t, |u(z, t)|) \in L^{q'}([-T, T]; \tilde{W}^{1,p'}(\mathbb{C}^n))$$

and the following inequalities hold:

$$(3.3) \quad \begin{aligned} & \|G(z, t, |u(z, t)|)\|_{L^{p',q'}} \\ & \leq CT^{\frac{q-q'}{qq'}} \|u(z, t)\|_{L^\infty([-T, T], \tilde{W}^{1,2}(\mathbb{C}^n))}^\alpha \|u(z, t)\|_{L^q([-T, T]; L^p(\mathbb{C}^n))}. \end{aligned}$$

$$(3.4) \quad \begin{aligned} & \|SG(z, t, |u(z, t)|)\|_{L^{p',q'}} \\ & \leq CT^{\frac{q-q'}{qq'}} \|u(z, t)\|_{L^\infty([-T, T], \tilde{W}^{1,2}(\mathbb{C}^n))}^\alpha \|u(z, t)\|_{L^q([-T, T]; \tilde{W}^{1,p}(\mathbb{C}^n))} \end{aligned}$$

Proof. Since $G(z, t, u(z, t)) = \psi(z, t, |u(z, t)|)u(z, t)$, and $\frac{q'}{q} + \frac{q-q'}{q} = 1$, an application of the Hölder's inequality in the t variable shows that for $q > 2$

$$(3.5) \quad \begin{aligned} & \|G(z, t, |u(z, t)|)\|_{L^{q'}([-T, T]; L^{p'}(\mathbb{C}^n))} \\ & \leq (2T)^{\frac{q-q'}{qq'}} \|\psi(z, t, |u(z, t)|)u(z, t)\|_{L^q([-T, T]; L^{p'}(\mathbb{C}^n))}. \end{aligned}$$

By an application of Hölders inequality in the z - variable, using $\frac{p'}{p} + \frac{\alpha p'}{p} = 1$, we see that for a.e. $t \in [-T, T]$

$$(3.6) \quad \begin{aligned} \|\psi(z, t, |u(z, t)|) u(z, t)\|_{L^{p'}(\mathbb{C}^n)} & \leq \|\psi(z, t, |u(z, t)|)\|_{L^{\frac{p}{\alpha}}(\mathbb{C}^n)} \|u(z, t)\|_{L^p(\mathbb{C}^n)} \\ & \leq C \|u(z, t)\|_{\tilde{W}^{1,2}(\mathbb{C}^n)}^\alpha \|u(z, t)\|_{L^p(\mathbb{C}^n)} \\ & \leq C \|u\|_{L^\infty([-T, T], \tilde{W}^{1,2})}^\alpha \|u(z, t)\|_{L^p(\mathbb{C}^n)}. \end{aligned}$$

where we used the condition (1.12) on ψ and Lemma 3.4 in the second inequality. Now taking L^q norm with respect to t on both sides, and substituting in the RHS of inequality (3.5) gives estimate (3.3).

To prove the inequality (3.4), we first observe that

$$(3.7) \quad \begin{aligned} L_j[\psi(x, y, t, |u|)u] &= \psi(x, y, t, |u|) L_j u \\ &+ u(\partial_4 \psi)(x, y, t, |u|) \Re \left(\frac{\bar{u}}{|u|} L_j u \right) + u(\partial_{x_j} \psi)(x, y, t, |u|) \end{aligned}$$

$$(3.8) \quad \begin{aligned} M_j[\psi(x, y, t, |u|)u] &= \psi(x, y, t, |u|) M_j u \\ &+ u(\partial_4 \psi)(x, y, t, |u|) \Re \left(\frac{\bar{u}}{|u|} M_j u \right) + u(\partial_{y_j} \psi)(x, y, t, |u|) \end{aligned}$$

Thus we see that for $S = L_j$ and M_j , $|SG|$ satisfies an inequality of the form

$$|SG| \leq |\psi(x, y, t, |u|) Su| + |\tilde{\psi}_1(x, y, t, |u|) Su| + |\tilde{\psi}_2(x, y, t, |u|) u|$$

where $\tilde{\psi}_1(x, y, t, |u|) = u\partial_4 \psi$ and $\tilde{\psi}_2(x, y, t, |u|) = u\partial_{x_j} \psi$ or $u\partial_{y_j} \psi$ depending on $S = L_j$ or M_j . Moreover, by assumption (1.12) on ψ , we have $|\tilde{\psi}_i(x, y, t, |u|)| \leq C|u|^\alpha$, $i = 1, 2$. Hence, using arguments as in inequalities (3.5) and (3.6), we get

$$\begin{aligned} & \|SG(z, t, |u(z, t)|)\|_{L^{q'}([-T, T]; L^{p'}(\mathbb{C}^n))} \\ & \leq CT^{\frac{q-q'}{qq'}} \|u(z, t)\|_{L^\infty([-T, T], \tilde{W}^{1,2}(\mathbb{C}^n))}^\alpha \times \\ & \quad (2\|Su(z, t)\|_{L^q([-T, T]; L^p(\mathbb{C}^n))} + \|u(z, t)\|_{L^q([-T, T]; L^p(\mathbb{C}^n))}) \\ & \leq CT^{\frac{q-q'}{qq'}} \|u(z, t)\|_{L^\infty([-T, T], \tilde{W}^{1,2}(\mathbb{C}^n))}^\alpha \|u(z, t)\|_{L^q([-T, T]; \tilde{W}^{1,p}(\mathbb{C}^n))} \end{aligned}$$

which is the inequality (3.4). \square

Proposition 3.8. Let $G(z, t, w)$ be as in (1.11) and α be as in assumption (1.12), $u \in L^\infty([-T, T]; \tilde{W}^{1,2}(\mathbb{C}^n)) \cap L^q([-T, T]; \tilde{W}^{1,p}(\mathbb{C}^n))$ where (q, p) is an admissible pair with $p = \alpha + 2$ and $q > 2$. Then for $|t| \leq T$, we have

$$(3.9) \quad \left\| \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s, u(z, s)) ds \right\|_{L^q([-T, T]; L^p(\mathbb{C}^n))} \\ \leq CT^{\frac{q-q'}{qq'}} \|u(z, t)\|_{L^\infty([-T, T]; \tilde{W}^{1,2})}^\alpha \|u(z, t)\|_{L^q([-T, T], L^p(\mathbb{C}^n))}$$

$$(3.10) \quad \left\| \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s, u(z, s)) ds \right\|_{L^q([-T, T]; \tilde{W}^{1,p}(\mathbb{C}^n))} \\ \leq CT^{\frac{q-q'}{qq'}} \|u(z, t)\|_{L^\infty([-T, T]; \tilde{W}^{1,2})}^\alpha \|u(z, t)\|_{L^q([-T, T], \tilde{W}^{1,p}(\mathbb{C}^n))}$$

Proof. It follows from Proposition 3.7, that $G(z, t, |u(z, t)|)$ and $SG(z, t, |u(z, t)|)$ are in $L^{q'}([-T, T]; L^{p'}(\mathbb{C}^n))$ for admissible pairs (q, p) with $p = \alpha + 2$, $q > 2$. Since (q, p) is admissible, by estimates (2.7) and (3.3), we get

$$(3.11) \quad \left\| \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s, u(z, s)) ds \right\|_{L^{p,q}} \\ \leq C \|G(z, s, |u(z, s)|)\|_{L^{p',q'}} \\ \leq CT^{\frac{q-q'}{qq'}} \|u(z, t)\|_{L^\infty([-T, T]; \tilde{W}^{1,2})}^\alpha \|u(z, t)\|_{L^{p,q}}$$

which is inequality (3.9).

Again by commutativity of L_j and M_j with $\int_0^t e^{-i(t-s)\mathcal{L}} ds$ and the fact that $SG(z, s, |u(z, s)|) \in L^{q'}([-T, T]; L^{p'}(\mathbb{C}^n))$, for $S = L_j, M_j, j = 1, 2, \dots, n$, we get as above

$$(3.12) \quad \left\| S \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s, u(z, s)) ds \right\|_{L^{p,q}} \\ \leq C \|SG(z, s, |u(z, s)|)\|_{L^{p',q'}} \\ \leq CT^{\frac{q-q'}{qq'}} \|u(z, t)\|_{L^\infty([-T, T]; \tilde{W}^{1,2})}^\alpha \|u(z, t)\|_{L^q([-T, T], \tilde{W}^{1,p}(\mathbb{C}^n))}$$

and the inequality (3.10) follows from the above two estimates. \square

Proposition 3.9. Let $G(z, t, w), \alpha, u$ be as in Proposition 3.8 and (q, p) an admissible pair with $p = \alpha + 2$ and $q > 2$. Then for $|t| \leq T$, we have

$$(3.13) \quad \left\| \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s, u(z, s)) ds \right\|_{L^\infty([-T, T]; L^2(\mathbb{C}^n))} \\ \leq CT^{\frac{q-q'}{qq'}} \|u(z, t)\|_{L^\infty([-T, T]; \tilde{W}^{1,2})}^\alpha \|u(z, t)\|_{L^q([-T, T], L^p(\mathbb{C}^n))}$$

$$\begin{aligned}
(3.14) \quad & \left\| \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s, u(z, s)) ds \right\|_{L^\infty([-T, T]; \tilde{W}^{1,2}(\mathbb{C}^n))} \\
& \leq CT^{\frac{q-q'}{qq'}} \|u(z, t)\|_{L^\infty([-T, T]; \tilde{W}^{1,2})}^\alpha \|u(z, t)\|_{L^q([-T, T], \tilde{W}^{1,p}(\mathbb{C}^n))}
\end{aligned}$$

Proof. The proof follows exactly as in Proposition 3.8, by using (2.8) instead of (2.7) in inequalities (3.11) and (3.12). \square

4. LOCAL EXISTENCE

In this section we prove local existence of solutions in the first order Sobolev space $\tilde{W}^{1,2}$. We follow Kato's approach using Strichartz estimates. The key step is to identify a subset in $L^\infty([-T, T]; \tilde{W}^{1,2}(\mathbb{C}^n))$, for a suitable T , where the operator \mathcal{H} is a contraction. We proceed as follows:

For given positive numbers T and M , consider the set $E = E_{T,M}$ given by

$$E = \left\{ u \in L^\infty([-T, T]; \tilde{W}^{1,2}) \cap L^q([-T, T], \tilde{W}^{1,p}) \mid \begin{array}{l} \|u\|_{L^\infty([-T, T], \tilde{W}^{1,2})} \leq M, \\ \|u\|_{L^q([-T, T], \tilde{W}^{1,p})} \leq M \end{array} \right\}$$

Introduce a metric on E , by setting

$$d(u, v) = \|u - v\|_{L^\infty([-T, T], L^2)} + \|u - v\|_{L^q([-T, T], L^p)}.$$

Proposition 4.1. (E, d) is a complete metric space.

Proof. Let $\{u_m\}$ be a Cauchy sequence in (E, d) . By passing to a subsequence if necessary, we see that $\{u_m(\cdot, t)\}$ is Cauchy in $L^2(\mathbb{C}^n, dz)$ as well as in $L^q(\mathbb{C}^n, dz)$ for almost all t . Going for a further subsequence and appealing to an almost everywhere convergence argument in z -variable, we conclude that they have the same almost everywhere limit, say $u(\cdot, t) \in L^q \cap L^2(\mathbb{C}^n)$ for almost all t . We need to show that $u \in L^\infty(I; \tilde{W}^{1,2}(\mathbb{C}^n)) \cap L^q(I; \tilde{W}^{1,p}(\mathbb{C}^n))$ with

$$\max\{\|u\|_{L^{2,\infty}}, \|L_j u\|_{L^{2,\infty}}, \|M_j u\|_{L^{2,\infty}}, j = 1, 2, \dots, n\} \leq M$$

and

$$\max\{\|u\|_{L^{p,q}}, \|L_j u\|_{L^{p,q}}, \|M_j u\|_{L^{p,q}}, j = 1, 2, \dots, n\} \leq M.$$

Let $S = L_j$ or M_j be as before and $\varphi \in C_c^\infty([-T, T] \times \mathbb{C}^n)$. Then for fixed $t \in [-T, T]$, using the pairing $\langle \cdot, \cdot \rangle_z$ in the z -variable, we see that

$$\begin{aligned}
|\langle u(\cdot, t), S^* \varphi(\cdot, t) \rangle_z| & \leq |\langle (u - u_m)(\cdot, t), S^* \varphi(\cdot, t) \rangle| + |\langle S u_m(\cdot, t), \varphi(\cdot, t) \rangle| \\
& \leq \|u(\cdot, t) - u_m(\cdot, t)\|_{L^p(\mathbb{C}^n, dz)} \|S^* \varphi(\cdot, t)\|_{L^{p'}(\mathbb{C}^n, dz)} \\
& \quad + \|S u_m(\cdot, t)\|_{L^p(\mathbb{C}^n, dz)} \|\varphi(\cdot, t)\|_{L^{p'}(\mathbb{C}^n, dz)}
\end{aligned}$$

Integrating with respect to t , and applying the Hölder's inequality in the t variable, this yields

$$|\langle Su, \varphi \rangle_{z,t}| \leq \|u - u_m\|_{L^{p,q}} \|S^* \varphi\|_{L^{p',q'}} + \|Su_m\|_{L^{p,q}} \|\varphi\|_{L^{p',q'}}$$

Since $u_m \in E$, $\|Su_m\|_{L^{p,q}} \leq M$, thus letting $m \rightarrow \infty$ we get

$$|\langle Su, \varphi \rangle_{z,t}| \leq \limsup_{m \rightarrow \infty} \|Su_m\|_{L^{p,q}} \|\varphi\|_{L^{p',q'}} \leq M \|\varphi\|_{L^{p',q'}}.$$

Taking supremum over all $\varphi \in C_c^\infty([-T, T] \times \mathbb{C}^n)$ with $\|\varphi\|_{L^{p',q'}} \leq 1$ this gives

$$\|Su\|_{L^q([-T, T]; L^p)} \leq M.$$

For the pair $(\infty, 2)$, take $\varphi \in C_c^\infty(\mathbb{C}^n)$, and by the same arguments as before

$$|\langle Su(\cdot, t), \varphi \rangle_z| \leq \limsup_{m \rightarrow \infty} \|Su_m(\cdot, t)\|_{L^2(\mathbb{C}^n)} \|\varphi\|_{L^2(\mathbb{C}^n)}$$

for almost every $t \in [-T, T]$. Taking supremum over all $\varphi \in C_c^\infty(\mathbb{C}^n)$ with $\|\varphi\|_{L^2} \leq 1$ this gives

$$\|Su(\cdot, t)\|_{L^2(\mathbb{C}^n)} \leq \limsup_{m \rightarrow \infty} \|Su_m\|_{L^{2,\infty}} \leq M.$$

Taking the essential supremum over $t \in [-T, T]$, we get the desired estimate. \square

Proof. (of Theorem 1.1) We employ the traditional method of using contraction mapping theorem. Let $u \in E$. From (1.10) and the estimates in Lemma 3.6 and Proposition 3.8, we see that for all admissible pair (q, p) ,

$$\begin{aligned} & \|\mathcal{H}u\|_{L^q([-T, T], \tilde{W}^{1,p})} \\ & \leq \|e^{-it\mathcal{L}} f(z)\|_{L^q([-T, T], \tilde{W}^{1,p})} + \left\| \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s, u(z, s)) ds \right\|_{L^q([-T, T], \tilde{W}^{1,p})} \\ & \leq C \|f\|_{\tilde{W}^{1,2}} \\ & \quad + C T^{\frac{q-q'}{qq'}} \|u(z, t)\|_{L^\infty([-T, T], \tilde{W}^{1,2})}^\alpha \|u(z, t)\|_{L^q([-T, T], \tilde{W}^{1,p}(\mathbb{C}^n))}. \end{aligned}$$

Clearly, this quantity is at most M , provided $T \leq \left(\frac{M - C\|f\|_{\tilde{W}^{1,2}}}{CM^{1+\alpha}} \right)^{\frac{qq'}{q-q'}}$.

The above inequalities are also valid for the pair $(\infty, 2)$ on LHS, by using Lemma 3.6 and Proposition 3.9. It follows that $\mathcal{H}u \in E = E_{T,M}$ provided

$$(4.1) \quad T \leq T_0 := \left(\frac{M - C\|f\|_{\tilde{W}^{1,2}}}{CM^{1+\alpha}} \right)^{\frac{qq'}{q-q'}}.$$

Now we show that for a given M , $\mathcal{H} : E_{T,M} \rightarrow E_{T,M}$ is a contraction for small T , i.e. for $T \leq \lambda T_0$ for some $\lambda < 1$.

Let $u, v \in E_{T,M}$ with T and M as in (4.1). By mean value theorem on ψ we see that

$$(4.2) \quad |G(z, s, u) - G(z, s, v)| \leq |u - v| \Psi(u, v)$$

where $\Psi(u, v) = (|w\partial_4\psi(x, y, s, w)| + |\psi(x, y, s, w)|)|_{w=\theta|u|+(1-\theta)|v|}$ for some $0 < \theta < 1$. Notice that in view of the condition (1.12) on ψ , $|\Psi(u, v)| \leq C(|u| \vee |v|)^\alpha$. Thus an application of Hölders inequality in z -variable, using the relation $\frac{p'}{p} + \frac{\alpha p'}{p} = 1$, followed by Sobolev embedding result (Lemma 3.4) gives

$$(4.3) \quad \begin{aligned} \|(u - v) \Psi(u, v)\|_{L^{p', q'}} &\leq C\|(|u| + |v|)^\alpha(u - v)\|_{L^{p', q'}} \\ &\leq CT^{\frac{q-q'}{qq'}} \|(|u| + |v|)\|_{L^\infty(I, \tilde{W}^{1,2})}^\alpha \|u - v\|_{L^{p, q}} \\ &\leq CT^{\frac{q-q'}{qq'}} M^\alpha \|u - v\|_{L^{p, q}} \end{aligned}$$

for $2 < q < \infty$. In view of the estimates (2.7), (4.2) and (4.3), the above lead to

$$(4.4) \quad \begin{aligned} \left\| \int_0^t e^{-i(t-s)\mathcal{L}} [G(u) - G(v)](z, s) ds \right\|_{L^{p, q}} &\leq C \|G(z, s, u) - G(z, s, v)\|_{L^{p', q'}} \\ &\leq CT^{\frac{q-q'}{qq'}} M^\alpha \|u - v\|_{L^{p, q}} \end{aligned}$$

for admissible pairs (q, p) . Similarly, using (2.8) instead of (2.7), we also get

$$(4.5) \quad \begin{aligned} \left\| \int_0^t e^{-i(t-s)\mathcal{L}} [G(u) - G(v)](z, s) ds \right\|_{L^{2, \infty}} &\leq CT^{\frac{q-q'}{qq'}} M^\alpha \|u - v\|_{L^{p, q}}. \end{aligned}$$

We choose $T = \lambda T_0$, for some $\lambda < 1$. Thus from (4.4) and (4.5), we have

$$(4.6) \quad d(\mathcal{H}(u), \mathcal{H}(v)) \leq 2C(\lambda T_0)^{\frac{q-q'}{qq'}} M^\alpha \|u - v\|_{L^{p, q}}.$$

Taking λ sufficiently small, we can make $k = 2C(\lambda T_0)^{\frac{q-q'}{qq'}} M^\alpha < 1$ and we get

$$d(\mathcal{H}(u), \mathcal{H}(v)) \leq k d(u, v).$$

This shows that $\mathcal{H} : E_{T,M} \rightarrow E_{T,M}$ is a contraction for $T = \lambda T_0$ and hence \mathcal{H} has a unique fixed point in $E = E_{\lambda T_0, M}$.

Note that we can fix a choice for M and T as follows: In view of the relation (4.1) between T_0 and M , when $f \neq 0$, we can choose $M = 2C\|f\|_{\tilde{W}^{1,2}}$ and $T = \lambda T_0 = \lambda C_1\|f\|_{\tilde{W}^{1,2}}^{-\alpha \frac{qq'}{q-q'}}$ with $C_1 = (2C)^{-(1+\alpha) \frac{qq'}{q-q'}}$, for any $\lambda < 1$, i.e.,

$$(4.7) \quad M = 2C\|f\|_{\tilde{W}^{1,2}} \text{ and } T < (2C)^{-(1+\alpha) \frac{qq'}{q-q'}} \|f\|_{\tilde{W}^{1,2}}^{-\alpha \frac{qq'}{q-q'}} \text{ if } f \neq 0.$$

When $f \equiv 0$, M can be any nonnegative number so that $T_0 = (CM^\alpha)^{\frac{qq'}{q'-q}}$ and $T = \lambda T_0$ will work for any $\lambda < 2^{\frac{qq'}{q'-q}}$. In particular, we can take

$$(4.8) \quad M = 1 \text{ and } T < (2C)^{\frac{qq'}{q'-q}} \text{ if } f \equiv 0.$$

Continuity: We will prove that $u \in C([-T, T]; \tilde{W}^{1,2}(\mathbb{C}^n))$. Let $t_m \rightarrow t$ and set $u_m = u(z, t_m)$, $S = L_j, M_j$ or the identity operator as before. Then

$$\begin{aligned} S(u_m - u) &= e^{-it_m \mathcal{L}} S f(z) - e^{-it \mathcal{L}} S f(z) - i \int_0^{t_m} e^{-i(t_m-s) \mathcal{L}} S G(z, s, u(z, s)) ds \\ &\quad + i \int_0^t e^{-i(t-s) \mathcal{L}} S G(z, s, u(z, s)) ds. \end{aligned}$$

Clearly $(e^{-it_m \mathcal{L}} - e^{-it \mathcal{L}}) S f(z) \rightarrow 0$ in $L^2(\mathbb{C}^n)$ as $t_m \rightarrow t$.

To deal with the other terms, we take $h \in L^2(\mathbb{C}^n)$ and estimate

$$\begin{aligned} &\left| \left\langle \int_0^t e^{-i(t_m-s) \mathcal{L}} S G(z, s, u(z, s)) ds - \int_0^t e^{-i(t-s) \mathcal{L}} S G(z, s, u(z, s)) ds, h \right\rangle \right| \\ &= \left| \int_0^t \langle e^{-i(t_m-s) \mathcal{L}} S G(z, s, u) - e^{-i(t-s) \mathcal{L}} S G(z, s, u) ds, h \rangle \right| \\ &= \left| \int_0^t \langle S G, (e^{i(t_m-s) \mathcal{L}} - e^{i(t-s) \mathcal{L}}) h \rangle ds \right| \\ &\leq \|S G\|_{L^{p', q'}} \|e^{-is \mathcal{L}} (e^{it_m \mathcal{L}} h - e^{it \mathcal{L}} h)\|_{L^{p, q}} \\ &\leq C \|S G\|_{L^{p', q'}} \|e^{it_m \mathcal{L}} h - e^{it \mathcal{L}} h\|_{L^2}. \end{aligned}$$

This shows that $\int_0^t e^{-i(t_m-s) \mathcal{L}} S G(z, s, u(z, s)) ds \rightarrow \int_0^t e^{-i(t-s) \mathcal{L}} S G(z, s, u(z, s)) ds$ weakly in $L^2(\mathbb{C}^n)$. This sequence is also bounded in L^2 :

$$\begin{aligned} &\left\| \int_0^t e^{-i(t_m-s) \mathcal{L}} S G(z, s, u(z, s)) ds \right\|_{L^2(\mathbb{C}^n)} \\ &= \left\| e^{-i(t_m-t) \mathcal{L}} \int_0^t e^{-i(t-s) \mathcal{L}} S G(z, s, u(z, s)) ds \right\|_{L^2(\mathbb{C}^n)} \\ &= \left\| \int_0^t e^{-i(t-s) \mathcal{L}} S G(z, s, u(z, s)) ds \right\|_{L^2(\mathbb{C}^n)} \end{aligned}$$

which is finite by Proposition 3.9. Thus we have the convergence in $L^2(\mathbb{C}^n)$:

$$\int_0^t e^{-i(t_m-s) \mathcal{L}} S G(z, s, u(z, s)) ds \rightarrow \int_0^t e^{-i(t-s) \mathcal{L}} S G(z, s, u(z, s)) ds.$$

Also since $\left\| \int_t^{t_m} e^{-i(t_m-s) \mathcal{L}} S G(z, s, u(z, s)) ds \right\|_{L^2} \leq C \|S G\|_{L^{q'}([t, t_m], L^{p'})} \rightarrow 0$ as $t_m \rightarrow t$, we conclude that $u(z, t_m) \rightarrow u(z, t)$ in $\tilde{W}^{1,2}$. \square

Remark 4.2. The above proof also shows that if we consider the initial value problem with an arbitrary initial time t_0 , then the solution exists on an interval

$[t_0 - T, t_0 + T]$ with T given by the same inequalities as in (4.7) and (4.8) but with $\|f\|_{\tilde{W}^{1,2}}$ replaced by $\|u(\cdot, t_0)\|_{\tilde{W}^{1,2}}$.

The following two results are used in the proof of Theorem 1.3.

Proposition 4.3. Let Φ be a continuous complex valued function on \mathbb{C} such that $|\Phi(w)| \leq C|w|^\alpha$ for $0 \leq \alpha < \frac{2}{n-1}$. Suppose $\{u_m\}$ be a sequence in $L^q([a, b], \tilde{W}^{1,p}) \cap L^\infty([a, b], \tilde{W}^{1,2})$, $p = 2 + \alpha$, $q \geq 2$, such that

$$\sup_{m \in \mathbb{N}} \|u_m\|_{L^\infty([a,b], \tilde{W}^{1,2})} \leq M < \infty.$$

If $u_m \rightarrow u$ in $L^q([a, b], L^p(\mathbb{C}^n))$ then $[\Phi(u_m) - \Phi(u)]Su \rightarrow 0$ in $L^{q'}([a, b], L^{p'}(\mathbb{C}^n))$, for $S = I, L_j, M_j; 1 \leq j \leq n$.

Proof. Since $u_m \rightarrow u$ in $L^q([a, b], L^p(\mathbb{C}^n))$, we can extract a subsequence still denoted by u_k such that

$$\|u_{k+1} - u_k\|_{L^q([a,b], L^p(\mathbb{C}^n))} \leq \frac{1}{2^k}$$

for all $k \geq 1$ and $u_k(z, t) \rightarrow u(z, t)$ a.e. Hence by continuity of Φ ,

$$(4.9) \quad [\Phi(u_k) - \Phi(u)]Su \rightarrow 0 \text{ for a.e. } (z, t) \in \mathbb{C}^n \times \mathbb{R}.$$

We establish the norm convergence by appealing to a dominated convergence argument in z and t variables successively.

Consider the function $H(z, t) = \sum_{k=1}^{\infty} |u_{k+1}(z, t) - u_k(z, t)|$. Clearly $H \in L^q([a, b], L^p(\mathbb{C}^n))$, since the above series converges absolutely in that space. Also for $l > k$, $|(u_l - u_k)(z, t)| \leq |u_l - u_{l-1}| + \cdots + |u_{k+1} - u_k| \leq H(z, t)$ hence $|u_k - u| \leq H$. This leads to the pointwise almost everywhere inequality

$$|u_k(z, t)| \leq |u(z, t)| + H(z, t) = v(z, t).$$

Hence

$$|[\Phi(u_k) - \Phi(u)]Su(z, t)|^{p'} \leq [v^\alpha + |u|^\alpha]Su(z, t)^{p'}.$$

Since $u, v \in L^q([a, b], L^p(\mathbb{C}^n))$ and $p = 2 + \alpha$, using Hölder's inequality with $\frac{p'}{p} + \frac{\alpha p'}{p} = 1$, we get

$$(4.10) \quad \int_{\mathbb{C}^n} |(v^\alpha + |u|^\alpha)Su(z, t)|^{p'} dz \leq (\|v(\cdot, t)\|_{L^p(\mathbb{C}^n)}^{\alpha p'} + \|u(\cdot, t)\|_{L^p(\mathbb{C}^n)}^{\alpha p'}) \|Su(\cdot, t)\|_{L^p(\mathbb{C}^n)}^{p'}.$$

Thus using dominated convergence theorem in z variable, we see that

$$(4.11) \quad \|[\Phi(u_k) - \Phi(u)]Su(\cdot, t)\|_{L^{p'}(\mathbb{C}^n)} \rightarrow 0$$

as $k \rightarrow \infty$ for a.e. t .

Again, in view of Lemma 3.4, and Hölder's inequality as above, we get

$$\begin{aligned}
& \|[\Phi(u_k) - \Phi(u)] Su(\cdot, t)\|_{L^{p'}(\mathbb{C}^n)} \\
& \leq C \left(\|u_k\|_{L^\infty([a,b], \tilde{W}^{1,2})}^\alpha + \|u\|_{L^\infty([a,b], \tilde{W}^{1,2})}^\alpha \right) \|Su(\cdot, t)\|_{L^p} \\
& \leq C(M^\alpha + \|u\|_{L^\infty([a,b], \tilde{W}^{1,2})}^\alpha) \|Su(\cdot, t)\|_{L^p}
\end{aligned}$$

Since $\|Su(\cdot, t)\|_{L^p(\mathbb{C}^n)} \in L^{q'}([a, b])$ and $q \geq 2$, an application of the Hölder's inequality in the t variable shows that

$$\int_a^b \|Su(\cdot, t)\|_{L^p(\mathbb{C}^n)}^{q'} dt \leq [b - a]^{\frac{q-q'}{q}} \|Su(\cdot, t)\|_{L^q([a,b], L^p(\mathbb{C}^n))}^{q'}$$

Hence a further application of dominated convergence theorem in (4.11) shows that $\|(\Phi(u_k) - \Phi(u)) Su\|_{L^{q'}([a,b], L^{p'})} \rightarrow 0$, as $k \rightarrow \infty$.

Thus we have shown that $[\Phi(u_{m_k}) - \Phi(u)] Su \rightarrow 0$ in $L^{q'}([a, b], L^{p'}(\mathbb{C}^n))$ whenever $u_m \rightarrow u$ in $L^q([a, b], L^p(\mathbb{C}^n))$. But the above arguments are also valid if we had started with any subsequence of u_m . It follows that any subsequence of $[\Phi(u_m) - \Phi(u)] Su$ has a subsequence that converges to 0 in $L^{q'}([a, b], L^{p'}(\mathbb{C}^n))$. From this we conclude that the original sequence $[\Phi(u_m) - \Phi(u)] Su$ converges to zero in $L^{q'}([a, b], L^{p'}(\mathbb{C}^n))$, hence the proposition. \square

Proposition 4.4. Let $\{f_m\}_{m \geq 1}$ be a sequence in $\tilde{W}^{1,2}(\mathbb{C}^n)$ such that $f_m \rightarrow f$ in $\tilde{W}^{1,2}(\mathbb{C}^n)$ as $m \rightarrow \infty$. Let u_m and u be the solutions corresponding to the initial data f_m and f respectively, at time $t = t_0$. Then there exists τ_0 , depending on $\|f\|_{\tilde{W}^{1,2}}$ such that $\|u_m - u\|_{L^\infty([t_0, t_0 + \tau_0], \tilde{W}^{1,2}(\mathbb{C}^n))} \rightarrow 0$.

Proof. Let $\epsilon > 0$ and $\tilde{\tau} < \lambda T_0$, where T_0 is as in (4.1) and λ as in (4.6). Since the time interval of existence is given by $\tilde{W}^{1,2}(\mathbb{C}^n)$ norm of the initial data and since $\|f_m\|_{\tilde{W}^{1,2}(\mathbb{C}^n)} \rightarrow \|f\|_{\tilde{W}^{1,2}(\mathbb{C}^n)}$ we can assume, by taking m large if necessary, that both the solutions u and u_m are defined on $[t_0, t_0 + \tilde{\tau}]$. Setting $G_m(z, t) = G(z, t, u_m(z, t))$, we have

$$(u_m - u)(z, t) = e^{-i(t-t_0)\mathcal{L}}(f_m - f)(z, t) - i \int_{t_0}^t e^{-i(t-s)\mathcal{L}}(G_m - G)(z, s) ds$$

for all $t \leq \tilde{\tau}$. In view of Lemma 3.5, with $S = I, L_j, M_j$, we also have

$$\begin{aligned}
S(u_m - u)(z, t) &= e^{-i(t-t_0)\mathcal{L}} S(f_m - f)(z, t) \\
&\quad - i \int_{t_0}^t e^{-i(t-s)\mathcal{L}} S(G_m - G)(z, s) ds.
\end{aligned}$$

Thus by estimates in Theorem 2.4, we see that for any $\tilde{\tau}_0 \leq \tilde{\tau}$

$$(4.12) \quad \|S(u_m - u)(z, t)\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p)} \leq C\|S(f_m - f)\|_2 \\ + C\|S(G_m - G)\|_{L^{q'}([t_0, t_0 + \tilde{\tau}_0], L^{p'})}$$

for $S = L_j, M_j$ and I , the identity operator for admissible pairs (q, p) .

First we consider the case $f \equiv 0$. In this case the solution $u \equiv 0$ since $\mathcal{H}(0) = 0$ and the fixed point of \mathcal{H} in E is unique. Thus by estimates in Theorem 2.4, we see that for any $\tilde{\tau}_0 \leq \tilde{\tau}$

$$(4.13) \quad \|Su_m(z, t)\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p)} \leq C\|Sf_m\|_2 \\ + C\|SG_m\|_{L^{q'}([t_0, t_0 + \tilde{\tau}_0], L^{p'})}$$

for $S = L_j, M_j$ and I , the identity operator. For $S = I$, using estimate (3.3), we get

$$\|u_m(z, t)\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p)} \leq C\|f_m\|_2 \\ + C\tilde{\tau}^{\frac{q-q'}{qq'}}\|u_m\|_{L^\infty([t_0, t_0 + \tilde{\tau}_0]; \tilde{W}^{1,2})}^\alpha\|u_m\|_{L^q([t_0, t_0 + \tilde{\tau}_0], \tilde{W}^{1,p})}$$

Now we observe that $\|u_m\|_{L^q([t_0, t_0 + \tilde{\tau}_0], \tilde{W}^{1,p})}$ is uniformly bounded. In fact by choice of $\tilde{\tau}_0$, from local existence theorem proved, we have $\|u_m\|_{L^q([t_0, t_0 + \tilde{\tau}_0], \tilde{W}^{1,p})} \leq M_m$ which is given by equations (4.7) and (4.8), for each m . Since

$$M_m = \begin{cases} 1 & \text{if } f_m = 0 \\ 2C\|f_m\|_{\tilde{W}^{1,2}} & \text{if } f_m \neq 0 \end{cases}$$

and $\|f_m\|_{\tilde{W}^{1,2}} \rightarrow \|f\|_{\tilde{W}^{1,2}} = 0$, we have $M_m \leq 1$ for large m .

Now choosing $\tilde{\tau}_0$ a value of $\tilde{\tau}$ small so that $C\tilde{\tau}^{\frac{q-q'}{qq'}} < \frac{1}{2}$, we see that

$$(4.14) \quad \|u_m\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p(\mathbb{C}^n))} \leq 2C\|f_m\|_{L^2(\mathbb{C}^n)} \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Thus in view of estimate (3.13) and estimate (2.8) in Theorem 2.4 we see that $u_m \rightarrow 0$ in $L^\infty([t_0, t_0 + \tilde{\tau}_0], L^2(\mathbb{C}^n))$.

Similarly, using the estimates (3.4) and (4.13), we see that $Su_m \rightarrow 0$ in $L^q([t_0, t_0 + \tilde{\tau}_0], L^p(\mathbb{C}^n))$. It follows from estimate (3.14) that $u_m \rightarrow 0$ in $L^\infty([t_0, t_0 + \tilde{\tau}_0], \tilde{W}^{1,2}(\mathbb{C}^n))$.

Now we consider the case $f \neq 0$. We choose m sufficiently large such that $\|f_m - f\|_{\tilde{W}^{1,2}} < \epsilon < \frac{1}{2}\|f\|_{\tilde{W}^{1,2}}$. Therefore we have $\|f_m\|_{\tilde{W}^{1,2}} \leq \frac{3}{2}\|f\|_{\tilde{W}^{1,2}}$ and hence again by (4.7), $M_m := 2C\|f_m\|_{\tilde{W}^{1,2}} \leq 2M := 4C\|f\|_{\tilde{W}^{1,2}}$. Now by (4.2), (4.3), and the fact that $M_m \leq 2M$ we get

$$(4.15) \quad \|G_m - G\|_{L^{q'}([t_0, t_0 + \tilde{\tau}], L^{p'}(\mathbb{C}^n))} \leq C\tilde{\tau}^{\frac{q-q'}{qq'}}\|f\|_{\tilde{W}^{1,2}}^\alpha\|u_m - u\|_{L^q([t_0, t_0 + \tilde{\tau}], L^p(\mathbb{C}^n))}.$$

This gives for the case $S = I$, the inequality

$$\begin{aligned} & \|u_m - u\|_{L^q([t_0, t_0 + \tilde{\tau}], L^p(\mathbb{C}^n))} \\ & \leq C\|f_m - f\|_{L^2} + C\tilde{\tau}^{\frac{q-q'}{qq'}}\|f\|_{\tilde{W}^{1,2}}^\alpha\|u_m - u\|_{L^q([t_0, t_0 + \tilde{\tau}], L^p(\mathbb{C}^n))} \end{aligned}$$

Now choosing $\tilde{\tau}_0$ a value of $\tilde{\tau}$ small so that $C\tilde{\tau}_0^{\frac{q-q'}{qq'}}\|f\|_{\tilde{W}^{1,2}}^\alpha < \frac{1}{2}$, we see that

$$(4.16) \quad \|u_m - u\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p(\mathbb{C}^n))} \leq C\|f_m - f\|_{L^2(\mathbb{C}^n)} < C\epsilon$$

for large m . Thus in view of Theorem 2.4, (4.15) and (4.16) we see that $u_m \rightarrow u$ in $L^\infty([t_0, t_0 + \tilde{\tau}_0], L^2(\mathbb{C}^n))$.

For $S = L_j, M_j$ using (3.7), (3.8) with the notation $\psi_m = \psi(z, t, |u_m(z, t)|)$, we have

$$\begin{aligned} (4.17) \quad S(G_m - G) &= \psi_m S(u_m - u) + (\psi_m - \psi)Su + (\partial_j \psi_m)(u_m - u) \\ &\quad + (\partial_j \psi_m - \partial_j \psi)u + (\partial_4 \psi_m)u_m \Re\left(\frac{\overline{u_m}}{|u_m|} S(u_m - u)\right) \\ &\quad + (\partial_4 \psi_m)u_m \Re\left(\frac{\overline{u_m}}{|u_m|} Su\right) - (\partial_4 \psi)u \Re\left(\frac{\overline{u}}{|u|} Su\right) \end{aligned}$$

where $\partial_j = \partial_{x_j}$ for $S = L_j$ and $\partial_j = \partial_{y_j}$ for $S = M_j$, $1 \leq j \leq n$.

Using the assumption (1.12) on ψ , Lemma 3.4, and the fact that $M_m \leq 2M$, similar computations as in Proposition 3.7 shows that

$$\begin{aligned} & \|\psi_m S(u_m - u)\|_{L^{q'}([t_0, t_0 + \tilde{\tau}_0], L^{p'})} \leq C\tilde{\tau}_0^{\frac{q-q'}{qq'}}\|f\|_{\tilde{W}^{1,2}}^\alpha\|S(u_m - u)\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p)} \\ & \|(\partial_j \psi_m)(u_m - u)\|_{L^{q'}([t_0, t_0 + \tilde{\tau}_0], L^{p'})} \leq C\tilde{\tau}_0^{\frac{q-q'}{qq'}}\|f\|_{\tilde{W}^{1,2}}^\alpha\|u_m - u\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p)} \\ & \|(\partial_4 \psi_m)u_m \Re\left(\frac{\overline{u_m}}{|u_m|} S(u_m - u)\right)\|_{L^{q'}([t_0, t_0 + \tilde{\tau}_0], L^{p'})} \\ & \leq C\tilde{\tau}_0^{\frac{q-q'}{qq'}}\|f\|_{\tilde{W}^{1,2}}^\alpha\|S(u_m - u)\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p)} \end{aligned}$$

Since $\|u_m - u\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p(\mathbb{C}^n))} \rightarrow 0$ and G is C^1 , so in view of the condition (1.12) on ψ and Lemma 4.3, the sequences $(\psi_m - \psi)Su$, $(\partial_j \psi_m - \partial_j \psi)u$ and $(\partial_4 \psi_m)u_m \Re\left(\frac{\overline{u_m}}{|u_m|} Su\right) - (\partial_4 \psi)u \Re\left(\frac{\overline{u}}{|u|} Su\right)$ converges to zero in $L^{q'}([t_0, t_0 + \tilde{\tau}_0], L^{p'})$ as $m \rightarrow \infty$. Hence for large m

$$\begin{aligned} & \|(\psi_m - \psi)Su\|_{L^{q'}([t_0, t_0 + \tilde{\tau}_0], L^{p'})} < \epsilon \\ & \|(\partial_j \psi_m - \partial_j \psi)u\|_{L^{q'}([t_0, t_0 + \tilde{\tau}_0], L^{p'})} < \epsilon \\ & \left\| (\partial_4 \psi_m)u_m \Re\left(\frac{\overline{u_m}}{|u_m|} Su\right) - (\partial_4 \psi)u \Re\left(\frac{\overline{u}}{|u|} Su\right) \right\|_{L^{q'}([t_0, t_0 + \tilde{\tau}_0], L^{p'})} < \epsilon. \end{aligned}$$

Using these estimates in (4.17), we get

$$\begin{aligned} \|S(G_m - G)\|_{L^{q'}([t_0, t_0 + \tilde{\tau}_0], L^{p'})} &\leq C\tilde{\tau}_0^{\frac{q-q'}{qq'}} \|f\|_{\tilde{W}^{1,2}}^\alpha \|S(u_m - u)\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p(\mathbb{C}^n))} \\ &\quad + C\tilde{\tau}_0^{\frac{q-q'}{qq'}} \|f\|_{\tilde{W}^{1,2}}^\alpha \|u_m - u\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p)} + 3\epsilon. \end{aligned}$$

Thus from estimates (4.12) and (4.16) we see that, for large m

$$\begin{aligned} \|S(u_m - u)\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p)} &\leq C\|S(f_m - f)\|_{L^2} + C\|S(G_m - G)\|_{L^{q'}([t_0, t_0 + \tilde{\tau}_0], L^{p'})} \\ &\leq C\tilde{\tau}_0^{\frac{q-q'}{qq'}} \|f\|_{\tilde{W}^{1,2}}^\alpha \|S(u_m - u)\|_{L^q([t_0, t_0 + \tilde{\tau}_0], L^p)} \\ &\quad + C^2\tilde{\tau}_0^{\frac{q-q'}{qq'}} \|f\|_{\tilde{W}^{1,2}}^\alpha \epsilon + (C + 3)\epsilon \end{aligned}$$

for large m . Now choose $\tau_0 < \tilde{\tau}_0$ so that

$$C\tau_0^{\frac{q-q'}{qq'}} \|f\|_{\tilde{W}^{1,2}}^\alpha \leq \frac{1}{2}$$

we see that $\|S(u_m - u)\|_{L^q([t_0, t_0 + \tau_0], L^p)} \leq (3C + 6)\epsilon$ for large m . Therefore $\|S(u_m - u)\|_{L^q([t_0, t_0 + \tau_0], L^p)} \rightarrow 0$ as $m \rightarrow \infty$. \square

Remark 4.5. The conclusions of the Proposition 4.4 is also valid for the left side interval $[t_0 - \tau_0, t_0]$.

Proof. (of Theorem 1.3) By local existence (Theorem 1), the solution exists in $C([-T, T] : \tilde{W}^{1,2}(\mathbb{C}^n))$. Now, if $\|u(\cdot, T)\|_{\tilde{W}^{1,2}(\mathbb{C}^n)} < \infty$, the argument in the proof of Theorem 1 can be carried out with T as the initial time, to extend the solution to the interval $[T, T_1]$. This procedure can be continued and we can get a sequence $\{T_j\}$ such that $T < T_1 < T_2 < \dots < T_n < \dots$ as long as $\|u(\cdot, T_j)\|_{\tilde{W}^{1,2}(\mathbb{C}^n)} < \infty$. Let $T_+ = \sup_j T_j$ so that the solution extends to $[0, T_+)$. In the same way we can extend the solution to the left side to the interval $(T_-, 0]$ to get a solution in $C((T_-, T_+), \tilde{W}^{1,2}(\mathbb{C}^n))$. This leads to the following proof of the

Blowup alternative: Suppose $T_+ < \infty$ and $\lim_{t \rightarrow T_+} \|u(z, t)\|_{\tilde{W}^{1,2}} = M_0 < \infty$. Then we can choose a sequence $t_j \uparrow T_+$ such that $\|u(z, t_j)\|_{\tilde{W}^{1,2}} \leq M_0$. From local existence and in view of Remark 4.2 we can choose $T_j = \frac{1}{10} C_1 \|u(\cdot, t_j)\|_{\tilde{W}^{1,2}}^{-\alpha \frac{qq'}{q-q'}}$ such that $u \in C([t_j - T_j, t_j + T_j], \tilde{W}^{1,2})$. Hence by assumption $T_j \geq \frac{1}{10} C_1 M_0^{-\alpha \frac{qq'}{q-q'}}$, a constant independent of t_j , for $q > 2$. Thus we can choose j so large that $t_j + T_j > T_+$, which contradicts maximality of T_+ . Hence if $T_+ < \infty$, $\lim_{t \rightarrow T_+} \|u(z, t)\|_{\tilde{W}^{1,2}} = \infty$. Similarly, we can show that $\lim_{t \rightarrow T_-} \|u(\cdot, t)\|_{\tilde{W}^{1,2}} = \infty$, if $T_- > -\infty$.

Uniqueness: Let (T_-, T_+) be the maximal interval about zero, such that the solution $u(x, t)$ exists in $C((T_-, T_+), \tilde{W}^{1,2})$ and in Theorem 1.1, we have already shown that the solution exists on $[-T, T] \subset (T_-, T_+)$. We first consider the subinterval $[0, T] \subset [0, T_+)$.

Suppose that there exist two solutions u and v of equations (1.6) and (1.7) on $[0, T_+)$. In particular for $\tau \in (T, T_+)$, we have

$$\begin{aligned} u(z, \tau) &= e^{-i(\tau-T)\mathcal{L}}u(z, T) - i \int_T^\tau e^{-i(\tau-s)\mathcal{L}}G(z, s, u(z, s))ds. \\ v(z, \tau) &= e^{-i(\tau-T)\mathcal{L}}v(z, T) - i \int_T^\tau e^{-i(\tau-s)\mathcal{L}}G(z, s, v(z, s))ds. \end{aligned}$$

Since the solution given by the contraction mapping is continuous and unique on $[0, T] \subset [0, T_+)$, we have $u(z, T) = v(z, T)$. Hence using an inequality as in (4.4), this leads to

$$\begin{aligned} \|u - v\|_{L^q([T, \tau], L^p(\mathbb{C}^n))} &= \left\| \int_T^\tau e^{-i(t-s)\mathcal{L}} (G(u) - G(v))(z, s) ds \right\|_{L^q([T, \tau], L^p(\mathbb{C}^n))} \\ &\leq C[\tau - T]^{\frac{q-q'}{qq'}} M_{T, \tau}^\alpha \|u - v\|_{L^q([T, \tau], L^p(\mathbb{C}^n))}. \end{aligned}$$

for all $\tau \in [T, T_+)$ where $M_{T, \tau} = \max\{\|u\|_{L^\infty([T, \tau], \tilde{W}^{1,2})}, \|v\|_{L^\infty([T, \tau], \tilde{W}^{1,2})}\}$. Since $u, v \in C((T_-, T_+), \tilde{W}^{1,2})$, we have $M_{T, \tau} < \infty$. In particular choose $\tilde{T} \in [T, T_+)$ such that $C|\tilde{T} - T|^{\frac{q-q'}{qq'}} M_{T, \tilde{T}}^\alpha = c < 1$, so that

$$0 \leq (1 - c)\|u - v\|_{L^q([T, \tilde{T}], L^p(\mathbb{C}^n))} \leq 0.$$

Hence $u = v$ on the larger interval $[0, \tilde{T}]$.

Now let $\theta = \sup\{\tilde{T} : 0 < \tilde{T} < T_+ : \|u - v\|_{L^q([0, \tilde{T}], L^p)} = 0\}$. If $\theta < T_+$, then by the above observation, $\|u - v\|_{L^q([0, \theta + \epsilon], L^p)} = 0$ for some $\epsilon > 0$, which contradicts the definition of θ . Thus we conclude that $\theta = T_+$, proving the uniqueness on $[0, T_+)$. Similarly one can show uniqueness on $(T_-, 0]$.

Stability: Let $\{f_m\}_{m \geq 1}$ be a sequence in $\tilde{W}^{1,2}(\mathbb{C}^n)$ such that $f_m \rightarrow f$ in $\tilde{W}^{1,2}$ as $m \rightarrow \infty$. Let u_m and u be the solutions corresponding to the initial data f_m and f respectively. Let (T_-, T_+) and (T_{m-}, T_{m+}) be maximal intervals for the solutions u and u_m respectively and $I \subset (T_-, T_+)$ be a compact interval.

The key idea is to extend the stability result proved in Proposition 4.4 to the interval I by covering it with finitely many intervals obtained by successive application of Proposition 4.4. This is possible provided u_m is defined on I , for all but finitely many m . In fact, we prove $I \subset (T_{m-}, T_{m+})$ for all but finitely many m .

We can assume that $0 \in I = [a, b]$, and give a proof by the method of contradiction. Suppose there exist infinitely many $T_{m_j+} \leq b$, let $c = \liminf T_{m_j+}$. Then for $\epsilon > 0$, $[0, c - \epsilon] \subset [0, T_{m_j+})$ for all m_j sufficiently large and u_{m_j} are defined on $[0, c - \epsilon]$.

By compactness, the stability result proved in Proposition 4.4 can be extended to the interval $[0, c - \epsilon]$ by covering it with finitely many intervals obtained by successive application of Proposition 4.4. Hence

$$\|u_{m_j}(\cdot, c - \epsilon)\|_{\tilde{W}^{1,2}} \rightarrow \|u(\cdot, c - \epsilon)\|_{\tilde{W}^{1,2}} \text{ as } j \rightarrow \infty.$$

Also by continuity we have

$$\|u(\cdot, c - \epsilon)\|_{\tilde{W}^{1,2}} \rightarrow \|u(\cdot, c)\|_{\tilde{W}^{1,2}} \text{ as } \epsilon \rightarrow 0.$$

Therefore, for any $\delta > 0$, we have

$$(4.18) \quad \|u_{m_j}(\cdot, c - \epsilon)\|_{\tilde{W}^{1,2}}^{-\alpha \frac{qq'}{q-q'}} > \delta \text{ whenever } \|u(\cdot, c)\|_{\tilde{W}^{1,2}}^{-\alpha \frac{qq'}{q-q'}} > \delta,$$

for sufficiently small ϵ and for all $j \geq j_0(\epsilon)$. Therefore by applying the local existence theorem, with $c - \epsilon$ as the initial time, we see that u_{m_j} extends to $[0, c - \epsilon + \frac{C_1}{10} \|u_{m_j}(\cdot, c - \epsilon)\|_{\tilde{W}^{1,2}}^{-\alpha \frac{qq'}{q-q'}}]$ for large j . Now choosing $\epsilon < \frac{C_1}{20} \delta$, we have by (4.18)

$$c - \epsilon + \frac{C_1}{10} \|u_{m_j}(\cdot, c - \epsilon)\|_{\tilde{W}^{1,2}}^{-\alpha \frac{qq'}{q-q'}} > c + \frac{C_1}{20} \delta \text{ for all } j \geq j_0(\epsilon).$$

It follows that $T_{m_j+} \geq c + \frac{C_1}{20} \delta$, hence contradicts the fact that $\liminf T_{m_j+} = c$.

Similarly we can show that $[a, 0] \subset (T_{m-}, 0]$ for all but finitely many m which completes the proof of stability. \square

5. APPENDIX

In this section we show the equivalence of the differential equation (1.6), (1.7) and the integral equation (1.9).

Lemma 5.1. Let $u \in L^\infty(I, \tilde{W}^{1,2}(\mathbb{C}^n)) \cap L^q(I, \tilde{W}^{1,p}(\mathbb{C}^n))$ and G as in (1.11), (1.12). Then u satisfies the non linear Schrödinger equation (1.6), with initial data (1.7) if and only if u satisfies the integral equation (1.9).

Proof. First observe that the following equalities

$$(5.1) \quad \frac{\partial}{\partial t}(e^{-it\mathcal{L}}f) = -i\mathcal{L}e^{-it\mathcal{L}}f$$

$$(5.2) \quad \frac{\partial}{\partial t} \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s) ds = -i\mathcal{L} \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s) ds + G(z, t)$$

are valid in the distribution sense for $f \in L^2(\mathbb{C}^n)$, $G \in L^{q'}(I, L^{p'})$. Using these we now show the equivalence of the initial value problem (1.6), (1.7) and the integral equations (1.9).

Note that $G(z, s, u(z, s)) \in L^{q'}(I, \tilde{W}^{1,p'}(\mathbb{C}^n))$, whenever $u \in L^\infty(I, \tilde{W}^{1,2}(\mathbb{C}^n)) \cap L^q(I, \tilde{W}^{1,p}(\mathbb{C}^n))$ by Proposition 3.7. Hence if such a u satisfy (1.9) then using (5.1) and (5.2), we conclude that u satisfy (1.6) and (1.7).

On the otherhand, if u satisfy (1.6) and (1.7) then the function v given by

$$v(z, t) = u(z, t) - e^{-it\mathcal{L}}f + i \int_0^t e^{-i(t-s)\mathcal{L}}G(z, s)ds.$$

satisfy

$$\begin{aligned} i\partial_t v(z, t) - \mathcal{L}v(z, t) &= 0, \\ v(z, 0) &= 0. \end{aligned}$$

The unique solution to this linear problem is given by $v(z, t) = e^{-it\mathcal{L}}v(z, 0) \equiv 0$ since $v(z, 0) = 0$. Therefore u satisfies (1.9).

Now we prove (5.1) and (5.2). Let $\phi \in C_c^\infty(\mathbb{C}^n \times I)$. Since I is an open interval, $\text{supp } \phi \subset A \times B$, for some compact set $A \subset \mathbb{C}^n$ and some compact interval $B \subset I$. Clearly,

$$\frac{\partial}{\partial t}(e^{-it\mathcal{L}}\bar{\phi}) = e^{-it\mathcal{L}}\frac{\partial}{\partial t}\bar{\phi} - e^{-it\mathcal{L}}i\mathcal{L}\bar{\phi}.$$

Also since $\phi(z, \cdot)$ has compact support in I for each z , $\int_I \frac{\partial}{\partial t}(e^{it\mathcal{L}}\phi)dt = 0$, hence

$$(5.3) \quad \int_I e^{-it\mathcal{L}}\frac{\partial}{\partial t}\bar{\phi}dt = \int_I e^{-it\mathcal{L}}i\mathcal{L}\bar{\phi}dt = i \overline{\int_I e^{it\mathcal{L}}\mathcal{L}\phi dt}.$$

Using this and the pairing $\langle f, \varphi \rangle = \int f\bar{\varphi}$, we see that

$$\begin{aligned} \int_{\mathbb{C}^n \times I} e^{-it\mathcal{L}}f(z) \frac{\partial}{\partial t}\overline{\phi(z, t)} dzdt &= \left\langle e^{-it\mathcal{L}}f, \frac{\partial}{\partial t}\phi \right\rangle = \left\langle f, e^{it\mathcal{L}}\frac{\partial}{\partial t}\phi \right\rangle \\ &= \langle f, -ie^{it\mathcal{L}}\mathcal{L}\phi \rangle = \langle i\mathcal{L}e^{-it\mathcal{L}}f, \phi \rangle. \end{aligned}$$

This proves (5.1) in the distribution sense.

To prove (5.2), choose a sequence $\{G_m\}$ in $C_c(A \times B)$ such that $G_m \rightarrow G$ in $L^{q'}(B, L^{p'}(A))$. Note that $G_m \in L^2(A \times B)$ hence,

$$\lim_{h \rightarrow 0} \frac{1}{h} [e^{-i(t+h-s)\mathcal{L}} - e^{-i(t-s)\mathcal{L}}] G_m(z, s) = -i\mathcal{L}e^{-i(t-s)\mathcal{L}}G_m(z, s)$$

and $\lim_{s \rightarrow t} e^{-i(t-s)\mathcal{L}} G_m(z, s) = G_m(z, t)$ where both the limits are taken in $L^2(\mathbb{C}^n)$ sense. Thus as an $L^2(\mathbb{C}^n)$ valued integral on I , we have

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_0^t e^{-i(t-s)\mathcal{L}} G_m(z, s) ds \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\int_0^{t+h} e^{-i(t+h-s)\mathcal{L}} G_m(z, s) ds - \int_0^t e^{-i(t-s)\mathcal{L}} G_m(z, s) ds \right) \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \int_0^{t+h} (e^{-i(t+h-s)\mathcal{L}} - e^{-i(t-s)\mathcal{L}}) G_m(z, s) ds \\
&\quad + \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} e^{-i(t-s)\mathcal{L}} G_m(z, s) ds \\
(5.4) \quad &= -i\mathcal{L} \int_0^t e^{-i(t-s)\mathcal{L}} G_m(z, s) ds + G_m(z, t)
\end{aligned}$$

Observe that $\int_0^t e^{-i(t-s)\mathcal{L}} G_m(z, s) ds \rightarrow \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s) ds$ in $L^q(B; L^p(A))$ as $m \rightarrow \infty$. This follows from estimate (2.7) since B is a bounded interval. Thus using (5.4), we see that

$$\begin{aligned}
\left\langle \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s) ds, \frac{\partial}{\partial t} \phi \right\rangle &= \lim_{m \rightarrow \infty} \left\langle \int_0^t e^{-i(t-s)\mathcal{L}} G_m(z, s) ds, \frac{\partial}{\partial t} \phi \right\rangle \\
&= \lim_{m \rightarrow \infty} \left\langle -\frac{\partial}{\partial t} \int_0^t e^{-i(t-s)\mathcal{L}} G_m(z, s) ds, \phi \right\rangle \\
&= \lim_{m \rightarrow \infty} \left\langle i\mathcal{L} \int_0^t e^{-i(t-s)\mathcal{L}} G_m(z, s) ds - G_m(z, t), \phi \right\rangle \\
&= \lim_{m \rightarrow \infty} \left\langle \int_0^t e^{-i(t-s)\mathcal{L}} G_m(z, s) ds, -i\mathcal{L} \phi \right\rangle - \langle G(z, t), \phi \rangle \\
&= - \left\langle \int_0^t e^{-i(t-s)\mathcal{L}} G(z, s) ds, i\mathcal{L} \phi \right\rangle - \langle G(z, t), \phi \rangle.
\end{aligned}$$

This shows that (5.2) holds in the distribution sense. \square

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